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Resolution of the nested hierarchy for rational $sl(n)$ models

T-D Albert, H Boos†, R Flume and K Ruhlig

Physikalisches Institut der Universität Bonn, Nußallee 12, D-53115 Bonn, Germany

E-mail: `t-albert@th.physik.uni-bonn.de`, `boos@th.physik.uni-bonn.de`, `flume@th.physik.uni-bonn.de` and `ruhlig@th.physik.uni-bonn.de`

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Abstract. We construct Drinfel'd twists for the rational $sl(n)$ *XXX* model giving rise to a completely symmetric representation of the monodromy matrix. We obtain a polarization-free representation of the pseudoparticle creation operators figuring in the construction of the Bethe vectors within the framework of the quantum inverse scattering method. This representation enables us to resolve the hierarchy of the nested Bethe ansatz for the $sl(n)$ invariant rational Heisenberg model. Our results generalize the findings of Maillet and Sanchez de Santos for $sl(2)$ models.

1. Introduction

In a seminal paper Maillet and Sanchez de Santos [1] revealed the uses of factorizing Drinfel'd twists [2] for inhomogeneous statistical spin-chain models for which the method of the algebraic Bethe ansatz is available. Those authors used the rational *XXX* and the trigonometric *XXZ* models as paradigms for their argument, being realized on tensor products of two-dimensional (fundamental) representations of the underlying group $sl(2)$. They showed that the similarity transformation provided by the Drinfel'd twist gives rise to a completely symmetric representation of the respective monodromy matrices and implies simplifying features in the new basis—to be described in detail below—for the various operators in the grid of the monodromy matrix.

The results of [1] have been generalized to any finite-dimensional irreducible representation of the Yangian $Y(sl(2))$ [3] and have been used to achieve substantial simplifications in the calculation of form factors [4], in the determination of thermodynamic quantities such as the spontaneous magnetization [5], and to solve the so-called quantum inverse problem [6, 7], that is, to express the local spin operators of the microscopic model through the operators figuring in the algebraic Bethe ansatz.

The most striking aspect of the results in [1] is, we think, related to the fact that no polarization clouds are attached to quasiparticle creation and annihilation operators in the basis in which the monodromy matrix is completely symmetric. This means, in terms of particle notation, that no virtual particle–antiparticle pairs are present in the wavevectors generated by the action of the creation operators to the ground state (the reference state of the Bethe ansatz), or in spin chain terminology that the creation and annihilation operators are constructed exclusively from local spin-raising and spin-lowering operators, respectively (that is, there are no compensating pairs of local raising and lowering spin operators). It was

† On leave of absence from the Institute of High Energy Physics, Protvino, Russia.

noted in [1] that this latter feature underlies the neat connection between the quantum spin chain models and their respective quasiclassical limits, which are Gaudin magnets [8], insofar as the appearance of the quasi-particle operators of the quantum models in the particular basis differs from the corresponding operators in the quasiclassical limit models only by a 'diagonal dressing' (see below).

This connection motivated us to attempt a generalization of the work of Maillet and Sanchez de Santos towards models based on higher-rank groups. We will deal here with the simplest conceivable extension in the form of the rational *XXX* model with $sl(n)$ as underlying group.

A notorious technical difficulty of integrable models with underlying higher-rank groups arises from the intricacies of the recursive procedure of the hierarchical Bethe ansatz [9]. It has been known for some time [10] that the recursion of the hierarchical ansatz can be resolved in the case of the quasiclassical limit of the rational models, i.e. the rational Gaudin magnets. Constructing the analogue of the factorizing twist of [1] for higher-rank models one may hope—in view of the affinity of the special basis rendered by the factorizing twist with the quasiclassical limit model—for an explicit resolution of the Bethe ansatz hierarchy. This will indeed be our main result for the spin model under consideration: an explicit representation of the $sl(n)$ Bethe wavevectors, solving therewith (for the wavevectors) the hierarchy.

The paper is organized as follows. Section 2 sets the notation, section 3 is devoted to the construction of the factorizing twist. In section 4 we give the expressions for the $sl(n)$ generators and for the operators contained in the monodromy matrix in the basis mediated by the factorizing twist. In section 5 we discuss the resolution of the Bethe hierarchy. Section 6 contains our conclusions. Some technical details are relegated to appendices.

2. Basic definitions and notation

Below we shall use much of the notation of [1, 4]. We consider the $sl(n)$ Yangian R-matrix to depend on a spectral parameter λ and a quantum deformation parameter η:

$$R_{12}(\lambda) = b(\lambda)\mathbb{1}_{12} + c(\lambda)P_{12} \tag{1}$$

where

$$b(\lambda) = \frac{\lambda}{\lambda+\eta} \qquad c(\lambda) = \frac{\eta}{\lambda+\eta}. \tag{2}$$

The matrix R_{12} is meant to represent a map $\mathbb{C}^n_{(1)} \otimes \mathbb{C}^n_{(2)} \to \mathbb{C}^n_{(1)} \otimes \mathbb{C}^n_{(2)}$ ($\mathbb{C}^n_{(1)} \cong \mathbb{C}^n_{(2)} \cong \mathbb{C}^n$) and P_{12} is the permutation operator acting in $\mathbb{C}^n_{(1)} \otimes \mathbb{C}^n_{(2)}$. Local spectral parameters attached to vector spaces $\mathbb{C}^n_{(i)}$ isomorphic to $\mathbb{C}^n$ will be called z_i. We will also use the notation

$$b_{ij} = b(z_i - z_j) \qquad c_{ij} = c(z_i - z_j). \tag{3}$$

It is well known that R-matrices defined by (1) satisfy the Yang–Baxter equation in vertex form,

$$R_{12}(z_1-z_2)R_{13}(z_1-z_3)R_{23}(z_2-z_3) = R_{23}(z_2-z_3)R_{13}(z_1-z_3)R_{12}(z_1-z_2) \tag{4}$$

and the unitarity relation

$$R_{12}R_{21} = \mathbb{1} \tag{5}$$

where $R_{ij} = R_{ij}(z_i - z_j)$ acts non-trivially on the tensor product $\mathbb{C}^n_{(i)} \otimes \mathbb{C}^n_{(j)}$.

Our convention for the matrix indices is as follows:

$$(Z)^{\gamma\,\delta}_{\beta\,\alpha} = (XY)^{\gamma\,\delta}_{\beta\,\alpha} = (X)^{\gamma\,\delta}_{j_1\,j_2}(Y)^{j_1\,j_2}_{\beta\,\alpha}. \tag{6}$$

With the notation $T_{0,23} = R_{03}R_{02}\ R_{0i} \equiv R_{0i}(z_i)$, where the index 0 refers to an auxiliary space $\mathbb{C}^n_{(0)}$, one may rewrite equation (4) in the form of a Faddeev–Zamolodchikov relation

$$R_{23}^{\sigma_{23}} T_{0,23} = T_{0,32} R_{23}^{\sigma_{23}} \tag{7}$$

with σ_{23} the transposition of space labels (2, 3).

We use here and subsequently a notation (which may not be in line with common use) that the labels in the upper row are permuted relative to lower indices according to the permutation inscribed, which reads in the example at hand as $(R^{\sigma_{23}})^{\alpha_2\,\alpha_3}_{\beta_3\,\beta_2}$.

It is straightforward to generalize equation (7) to an N-fold tensor product of spaces.

With the definition $T_{0,1\ldots N} = R_{0N}\ldots R_{01}$ the generalization reads

$$R^{\sigma}_{1\ldots N} T_{0,1\ldots N} = T_{0,\sigma(1)\ldots\sigma(N)} R^{\sigma}_{1\ldots N} \tag{8}$$

where σ is now an element of the symmetric group S_N and $R^{\sigma}_{1\ldots N}$ denotes a product of R-matrices occurring in (7), the product corresponding to a decomposition of σ into elementary transpositions.

The order of the upper matrix indices α_i of R^{σ} reads according to the above prescription as follows:

$$\left(R^{\sigma}_{1\ldots N}\right)^{\alpha_{\sigma(N)}\ldots\alpha_{\sigma(1)}}_{\beta_N\ldots\beta_1}. \tag{9}$$

Equation (8) implies the composition law (note the difference to the composition law used in [1])

$$R^{\sigma'\sigma}_{1\ldots N} = R^{\sigma}_{\sigma'(1)\ldots\sigma'(N)} R^{\sigma'}_{1\ldots N} \tag{10}$$

for a product of two elements in S_N. The factor $R^{\sigma'}_{\sigma(1)\ldots\sigma(N)}$ on the right-hand side of equation (10) satisfies the relation

$$R^{\sigma'}_{\sigma(1)\ldots\sigma(N)} T_{0,\sigma(1)\ldots\sigma(N)} = T_{0,\sigma\sigma'(1)\ldots\sigma\sigma'(N)} R^{\sigma'}_{\sigma(1)\ldots\sigma(N)}. \tag{11}$$

3. The F-matrix and some of its properties

The starting point of [1] is the Drinfel'd factorizing twists of the elementary $sl(2)$ R-matrix:

$$R_{12} = F_{21}^{-1} F_{12}$$

where F_{12} is given by formula (90) of [1]

$$F_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c(z_1 - z_2) & b(z_1 - z_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

The generalization of this formula to the $sl(n)$ case is of the form

$$F_{12} = \sum_{n\geqslant\alpha_2\geqslant\alpha_1} P^1_{\alpha_1} P^2_{\alpha_2}\, \mathbb{1}_{12} + \sum_{n\geqslant\alpha_1>\alpha_2} P^1_{\alpha_1} P^2_{\alpha_2} R^{\sigma_{12}}_{12}. \tag{13}$$

Here $[P^i_{\alpha}]_{k,l} = \delta_{k,\alpha}\delta_{l,\alpha}$ is the projector on the α component acting in ith space.

Generalizing this factorization matrix to the N-site problem one has to satisfy at least three properties for the F-matrix (see [1, 4]):

(A) factorization, that is

$$F_{\sigma(1)\ldots\sigma(N)}(z_{\sigma(1)}, \ldots, z_{\sigma(N)}) R^{\sigma}_{1\ldots N}(z_1, \ldots, z_N) = F_{1\ldots N}(z_1, \ldots, z_N) \tag{14}$$

for any permutation $\sigma \in S_N$;

(B) lower-triangularity;
(C) non-degeneracy.

Proposition 3.1. *The following expression for the F-matrix:*

$$F_{1\ldots N} = \sum_{\sigma\in S_N} \sum^{*}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}} \prod_{i=1}^{N} P^{\sigma(i)}_{\alpha_{\sigma(i)}} R^{\sigma}_{1\ldots N}(z_1,\ldots,z_N) \tag{15}$$

satisfies the properties A, B and C. The sum $\sum^*$ is to be taken over all non-decreasing sequences of the labels $\alpha_{\sigma(i)}$ which are increasing at places where the permuted index is decreasing ($\sigma(i+1) < \sigma(i)$), namely, labels α_i should satisfy one of two inequalities for each pair of neighbouring spaces labels:

$$\begin{aligned} &\alpha_{\sigma(i+1)} \geqslant \alpha_{\sigma(i)} && \textit{if} \quad \sigma(i+1) > \sigma(i) \\ &\alpha_{\sigma(i+1)} > \alpha_{\sigma(i)} && \textit{if} \quad \sigma(i+1) < \sigma(i). \end{aligned} \tag{16}$$

Proof. First of all let us note that the lower-triangularity can be traced back to the form of the elementary R-matrix using the definition of F, equation (15). Indeed, the ordering (16) just corresponds to the lower-triangularity of the matrix F. Non-degeneracy follows from the lower-triangularity and the fact that all diagonal elements are non-zero. Apart from that we shall give below the explicit form of F^{-1}.

To prove the factorization property A let us, as above, represent the arbitrary permutation σ in the form the composition of k elementary transpositions σ_i, i.e.

$$\sigma = \sigma_1 \ldots \sigma_k.$$

The important structural feature of equation (15) is that it can be decomposed stepwise into elementary transpositions:

$$\begin{aligned} F_{\sigma(1)\ldots\sigma(N)}\, R^{\sigma}_{1\ldots N} &= F_{\sigma_1\sigma_2\ldots\sigma_k(1,\ldots,N)}\, R^{\sigma_k}_{\sigma_1\sigma_2\ldots\sigma_{k-1}(1,\ldots,N)}\, R^{\sigma_{k-1}}_{\sigma_1\sigma_2\ldots\sigma_{k-2}(1,\ldots,N)} \cdots R^{\sigma_1}_{1\ldots N} \\ &= F_{\sigma_1\sigma_2\ldots\sigma_{k-1}(1,\ldots,N)}\, R^{\sigma_{k-1}}_{\sigma_1\sigma_2\ldots\sigma_{k-2}(1,\ldots,N)}\, R^{\sigma_{k-2}}_{\sigma_1\sigma_2\ldots\sigma_{k-3}(1,\ldots,N)} \cdots R^{\sigma_1}_{1\ldots N} \\ &= \cdots\cdots\cdots = F_{\sigma_1(1,\ldots,N)}\, R^{\sigma_1}_{1\ldots N} = F_{1\ldots N} \end{aligned}$$

where the composition law (10) was used. So we have to prove equation (15) for elementary transpositions only.

Let σ_i be the elementary transposition $\{i, i+1\} \to \{i+1, i\}$. We consider the product $F_{1\ldots i+1\, i\ldots N} R^{\sigma_i}_{1\ldots N}$. With the help of equations (15) and (10) we obtain

$$\begin{aligned} F_{1\ldots i+1\, i\ldots N} R^{\sigma_i}_{1\ldots N} &= F_{\sigma_i(1\ldots i\; i+1\ldots N)} R^{\sigma_i}_{1\ldots N} \\ &= \sum_{\sigma\in S_N} \sum^{*(i)}_{\alpha_{\sigma_i\sigma(1)}\ldots\alpha_{\sigma_i\sigma(N)}} \prod_{j=1}^{N} P^{\sigma_i\sigma(j)}_{\alpha_{\sigma_i\sigma(j)}} R^{\sigma}_{\sigma_i(1,\ldots,N)} R^{\sigma_i}_{1\ldots N} \\ &= \sum_{\sigma\in S_N} \sum^{*(i)}_{\alpha_{\sigma_i\sigma(1)}\ldots\alpha_{\sigma_i\sigma(N)}} \prod_{j=1}^{N} P^{\sigma_i\sigma(j)}_{\alpha_{\sigma_i\sigma(j)}} R^{\sigma_i\sigma}_{1\ldots N} \end{aligned} \tag{17}$$

with $\sum^{*(i)}$ being defined by the restricting conditions

$$\begin{aligned} &\alpha_{\sigma_i\sigma(j+1)} \geqslant \alpha_{\sigma_i\sigma(j)} && \text{if} \quad \sigma(j+1) > \sigma(j) \\ &\alpha_{\sigma_i\sigma(j+1)} > \alpha_{\sigma_i\sigma(j)} && \text{if} \quad \sigma(j+1) < \sigma(j). \end{aligned} \tag{18}$$

(It may be helpful to bear in mind that the ordering prescription has to be executed according to the shifted labels $\widetilde{j} = \sigma_i(j)$.) Substituting in (17) $\tilde{\sigma}$ for $\sigma_i\sigma$ one arrives at

$$F_{1\ldots i+1\, i\ldots N} R^{\sigma_i}_{1\ldots N} = \sum_{\tilde{\sigma}\in S_N} \sum^{*}_{\alpha_{\tilde{\sigma}(1)}\ldots\alpha_{\tilde{\sigma}(N)}} \prod_{j=1}^{N} P^{\tilde{\sigma}(j)}_{\alpha_{\tilde{\sigma}(j)}} R^{\tilde{\sigma}}_{1\ldots N} \tag{19}$$

with the defining restrictions of $\sum^*$ now of the form

$$\begin{aligned} &\alpha_{\tilde{\sigma}(j+1)} \geqslant \alpha_{\tilde{\sigma}(j)} &&\text{if} \quad \sigma_i\tilde{\sigma}(j+1) > \sigma_i\tilde{\sigma}(j) \\ &\alpha_{\tilde{\sigma}(j+1)} > \alpha_{\tilde{\sigma}(j)} &&\text{if} \quad \sigma_i\tilde{\sigma}(j+1) < \sigma_i\tilde{\sigma}(j) \end{aligned} \tag{20}$$

which has a slightly different appearance in comparison to (16). Elementary combinatorial considerations lead to the conclusion that the stipulations (16) and (20) give the same result as long as $\sigma^{-1}(i)$ and $\sigma^{-1}(i+1)$ do not happen to be on neighbouring places, that is if not

$$\sigma^{-1}(i) = \sigma^{-1}(i+1) \pm 1. \tag{21}$$

If (21) holds we have to appeal to the specific form of the R-matrix to complete the argument.

Comparing the right-hand side of equation (19) in connection with (20) to the right-hand side of equation (15) in connection with (16), one notes that a discrepancy is certainly excluded if the strict inequality is implied in the step from $\sigma^{-1}(i)$ to $\sigma^{-1}(i+1)$ (if $\sigma^{-1}(i+1)$ is larger than $\sigma^{-1}(i)$), or from $\sigma^{-1}(i+1)$ to $\sigma^{-1}(i)$ if the reversed order is assumed. However, for equal group labels at the two neighbouring places in question, the representation of the additional transposition of i and $i+1$ in (15) as compared with (19) has no effect, since it supplies a unit factor due to the projectors.

This completes the proof of the proposition. □

Remark. The most general matrix $\tilde{F}$ satisfying the above conditions A and C differs from the special solution of the preceding theorem by a non-degenerate, completely symmetric matrix factor [1],

$$\begin{aligned} &\tilde{F}_{1\ldots N}(z_1,\ldots,z_N) = X_{1\ldots N}(z_1,\ldots,z_N) F_{1\ldots N}(z_1,\ldots,z_N) \\ &X_{1\ldots N}(z_1,\ldots,z_N) = X_{\sigma(1)\ldots\sigma(N)}(z_{\sigma(1)},\ldots,z_{\sigma(N)}) \qquad \forall\sigma \in S_N. \end{aligned}$$

Indeed, it is easy to see that $\tilde{F}$, together with F, satisfies the factorization equation (14). Conversely, suppose that both F and $\tilde{F}$ satisfy (14). It follows that

$$F^{-1}_{\sigma(1)\ldots\sigma(N)} F_{1\ldots N} = \tilde{F}^{-1}_{\sigma(1)\ldots\sigma(N)} \tilde{F}_{1\ldots N}$$

and therefore

$$F_{1\ldots N} \tilde{F}^{-1}_{1\ldots N} = F_{\sigma(1)\ldots\sigma(N)} \tilde{F}^{-1}_{\sigma(1)\ldots\sigma(N)}.$$

Hence it follows that

$$X_{1\ldots N} = F_{1\ldots N} \tilde{F}^{-1}_{1\ldots N}$$

is non-degenerate and completely symmetric and transforms $\tilde{F}$ into F, $X_{1\ldots N}\tilde{F}_{1\ldots N} = F_{1\ldots N}$.

Furthermore, we need the inverse operator F^{-1}. To find its expression we have to prove the following

Proposition 3.2. *The operator F^* defined by the formula*

$$F^*_{1\ldots N} = \sum_{\sigma\in S_N} \sum^{**}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}} R^{(t)\,\sigma}_{1\ldots N}(z_1,\ldots,z_N) \prod_{i=1}^{N} P^{\sigma(i)}_{\alpha_{\sigma(i)}} \tag{22}$$

with the shorthand notation

$$R^{(t)\,\sigma}_{1\ldots N} \equiv R^{\sigma^{-1}}_{\sigma(1,\ldots,N)} \tag{23}$$

and $\sum^{**}$ *is taken over all possible* α_i *which satisfy one of the two inequalities for each neighbouring pair of spaces* i *and* $i+1$*:*

$$\begin{aligned} &\alpha_{\sigma(i+1)} \leqslant \alpha_{\sigma(i)} && \text{if} \quad \sigma(i+1) < \sigma(i) \\ &\alpha_{\sigma(i+1)} < \alpha_{\sigma(i)} && \text{if} \quad \sigma(i+1) > \sigma(i) \end{aligned} \tag{24}$$

satisfy the relation

$$F_{1\ldots N} F^*_{1\ldots N} = \prod_{i<j} \Delta_{ij} \tag{25}$$

where the diagonal matrix

$$[\Delta_{ij}]^{\beta_i,\beta_j}_{\alpha_i,\alpha_j} = \delta_{\alpha_i\beta_i}\delta_{\alpha_j\beta_j} \begin{cases} 1 & \text{if} \quad \alpha_i = \alpha_j \\ b_{ij} & \text{if} \quad \alpha_i > \alpha_j \\ b_{ji} & \text{if} \quad \alpha_j > \alpha_i \end{cases} \tag{26}$$

acts in the pair of spaces i *and* j*.*

Proof. Taking into account the conditions (16) and (24) in sums $\sum^*$ and $\sum^{**}$ of the expressions (15) and (22), respectively, one can write down the expression for the product $F_{1\ldots N} F^*_{1\ldots N}$ in the following form:

$$\begin{aligned} F_{1\ldots N} F^*_{1\ldots N} &= \sum_{\sigma\in S_N} \sum_{\sigma'\in S_N} \sum^*_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}} \sum^{**}_{\beta_{\sigma'(1)}\ldots\beta_{\sigma'(N)}} \prod_{i=1}^N P^{\sigma(i)}_{\alpha_{\sigma(i)}} R^{\sigma}_{1\ldots N} R^{\sigma'^{-1}}_{\sigma'(1)\ldots\sigma'(N)} \prod_{i=1}^N P^{\sigma'(i)}_{\beta_{\sigma'(i)}} \\ &= \sum_{\sigma\in S_N} \sum_{\sigma'\in S_N} \sum^*_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}} \sum^{**}_{\beta_{\sigma'(1)}\ldots\beta_{\sigma'(N)}} \prod_{i=1}^N P^{\sigma(i)}_{\alpha_{\sigma(i)}} R^{\sigma'^{-1}\sigma}_{\sigma'(1,\ldots,N)} \prod_{i=1}^N P^{\sigma'(i)}_{\beta_{\sigma'(i)}} \qquad (27) \\ &= \sum_{\sigma\in S_N} \sum^*_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}} \prod_{i=1}^N P^{\sigma(i)}_{\alpha_{\sigma(i)}} R^{\overline{\sigma}}_{\sigma(N,\ldots,1)} \prod_{i=1}^N P^{\sigma(i)}_{\alpha_{\sigma(i)}} \qquad (28) \end{aligned}$$

where the permutation $\overline{\sigma}$ reverses the order of the labels:

$$\overline{\sigma}(1,\ldots,N) = (N,\ldots,1).$$

In the line above (27) we have inserted the definitions of F and F^*, equations (15) and (22), respectively. Equality (27) is obtained by applying the composition rule (10). To prove equality (28) we note first of all that any matrix R^σ provides maps such that the sets of $sl(n)$ labels of the incoming and outgoing states are connected by a permutation. (This property is easily verified for matrices R^σ corresponding to elementary transpositions and it is preserved under the composition of several transpositions.) However, the labels $\{\alpha_{\sigma(i)}\}$ represent, according to the prescription (16), a non-decreasing series (in (i)) of labels, whereas the $\{\beta_{\sigma'(i)}\}$ (being related to $\{\alpha_{\sigma(i)}\}$ by a permutation) are according to (24) a non-increasing series. For these two requirements to be fulfilled the equalities

$$\beta_{\sigma'(N)} = \alpha_{\sigma(1)} \quad \ldots \quad \beta_{\sigma'(1)} = \alpha_{\sigma(N)} \tag{29}$$

are necessary.

Let us assume momentarily that all the labels $\beta_{\sigma'(i)}$ (and hence the $\alpha_{\sigma(i)}$) are different from each other. We want to show that equation (29) implies the equality

$$\sigma\,\bar{\sigma} = \sigma' \tag{30}$$

for the matrix element

$$\left(R^{\sigma'^{-1}\sigma}_{\sigma'(1,\ldots,N)}\right)^{\alpha_{\sigma(N)}\ldots\alpha_{\sigma(1)}}_{\beta_{\sigma'(N)}\ldots\beta_{\sigma'(1)}}$$

to be non-vanishing. Viewing $R^{\sigma'^{-1}\sigma}_{\sigma'(1,\ldots,N)}$ as a product of elementary R-matrices one observes that the group label $\beta_{\sigma'(N)} = \alpha_{\sigma(1)}$ can be transported from the lower left-hand corner to the upper right-hand corner only if the space labels $\sigma'(N)$ and $\sigma(1)$ are identical. Assume, in contrast, that $\sigma'(N)$ is identical to some other element $\sigma(x) \neq \sigma(1)$. It would follow that the group label $\beta_{\sigma'(N)}$ could appear in the upper row only at the place with space label $\sigma(x)$ or on the left-hand side of it. (This restriction on the flow of group labels is a straightforward consequence of the form of the elementary R-matrix, equation (1).) We conclude that we have indeed to identify $\sigma'(N)$ with $\sigma(1)$ to obtain a non-vanishing matrix element of R. The identification of $\sigma'(N-1)$ with $\sigma(2)$, etc follows analogously from equation (30).

A glance at (16) and (24) confirms that equation (30) remains valid under general circumstances, i.e. if some group labels $\beta_{\sigma'(i)}$ and therefore $\alpha_{\sigma(j)}$ occur repeatedly, since the order of the space labels attached to the same group label is uniquely specified by these prescriptions.

One deduces from (28) that FF^* is a diagonal matrix. A simple calculation leads to the expression for the diagonal elements quoted in equation (26). (The product appearing on the right-hand side of equation (25) is related to $\bar{\sigma}$ as the latter is a maximal element of S_N and as such is representable as a product of $N(N-1)/2$ elementary transpositions. Each transposition is reflected in one factor of the product in equation (25).)

This completes the proof of proposition 3.2. □

We obtain from the formula (25) the expression for $F^{-1}_{1\ldots N}$:

$$F^{-1}_{1\ldots N} = F^*_{1\ldots N} \prod_{i<j} \Delta^{-1}_{ij}. \tag{31}$$

For the case of the $sl(2)$ Yangian the formula (31) corresponds to the result of proposition 4.6 of [1].

4. $sl(n)$ generators and the monodromy matrix in the F-basis

We will first determine the simple root $sl(n)$ generators $\tilde{E}_{\alpha,\alpha\pm1} = F_{1\ldots N} E_{\alpha\pm\alpha+1} F^{-1}_{1\ldots N}$ and the element $\tilde{T}_{nn} = F_{1\ldots N} T_{nn} F^{-1}_{1\ldots N}$ of the monodromy matrix. The remaining $sl(n)$ generators can then be obtained from the simple ones through multiple commutators. The examination of the full algebra can be found in appendix A. One may exploit the $sl(n)$ invariance of the monodromy matrix (with respect to its combined action in the quantum spaces and the auxiliary space, see, e.g., [11]) to derive expressions for all elements $\tilde{T}_{\alpha\beta}$ given $\tilde{T}_{nn}$ and the $sl(n)$ generators.

One has, in particular, the relation

$$\tilde{T}_{n\alpha} = \left[\tilde{E}_{\alpha,n}, \tilde{T}_{nn}\right]. \tag{32}$$

The left-hand side of the latter equation originates from the action of the $sl(n)$ generator in the auxiliary space, whereas the right-hand side evidently reflects the corresponding action in the quantum space.

We will content ourselves to derive the explicit form of $\tilde{T}_{n\alpha}$ using equation (32), since this is all we need to build $sl(n)$ Bethe wavevectors.

The simple root generators in the new basis differ from those in the original basis by a diagonal dressing factor. We have

Proposition 4.1.

$$\tilde{E}_{\gamma,\gamma\pm 1} = \sum_{i=1}^{N} E^{(i)}_{\gamma,\gamma\pm 1} \otimes_{j\neq i} G^{\pm\gamma}(i,j)_{[j]} \tag{33}$$

where

$$\begin{aligned} G^{\gamma}(i,j)_{k,l} &= \delta_{kl} \begin{cases} b_{ij}^{-1} & \text{if} \quad k=\gamma \\ 1 & \text{otherwise} \end{cases} \\ G^{-\gamma}(i,j)_{k,l} &= \delta_{kl} \begin{cases} b_{ji}^{-1} & \text{if} \quad k=\gamma+1 \\ 1 & \text{otherwise.} \end{cases} \end{aligned} \tag{34}$$

Proof. Equations (33) and (34) specialized to the rational $sl(2)$ case have been presented in propositions 5.1 and 5.2 of [1]. The proof of these equations for the $sl(n)$ model with arbitrary n can be reduced to that of the $sl(2)$ model. One has to note for this purpose that due to the $sl(n)$ invariance of the elementary R-matrices, one obtains the vanishing result

$$\left[R^{\sigma}_{1\ldots N}, E_{\alpha,\alpha\pm 1}\right] = 0 \tag{35}$$

for any permutation $\sigma \in S_N$.

This allows us to write (cf equation (27))

$$\tilde{E}_{\gamma,\gamma\pm 1} = \sum_{\sigma\in S_N}\sum_{\sigma'\in S_N}\sum^{*}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}\sum^{**}_{\beta_{\sigma'(1)}\ldots\beta_{\sigma'(N)}}\prod_{i=1}^{N} P^{\sigma(i)}_{\alpha_{\sigma(i)}} E_{\gamma,\gamma\pm 1} R^{\sigma'^{-1}\sigma}_{\sigma'(1,\ldots,N)} \prod_{i=1}^{N} P^{\sigma'(i)}_{\beta_{\sigma'(i)}} \prod_{i<j}\Delta_{ij}. \tag{36}$$

The collapse of the double sum $\sum_{\sigma,\sigma'}$ into a single sum proceeds here along the same pattern as above (in the transition from equation (27) to equation (28)). One has to further note that group indices γ and $(\gamma+1)$ ($(\gamma-1)$, respectively) only occur in neighbouring positions concerning ingoing and outgoing matrix indices because of the monotonicity prescription incorporated into the sums $\sum^{*}$ and $\sum^{**}$, respectively. The rearrangement of the neighbouring labels γ and $(\gamma+1)$ ($(\gamma-1)$, respectively) goes on according to $sl(2)$ rules and produces the result quoted in equation (34) and in [1]. Rearrangements involving group indices different from γ and $(\gamma+1)$ ($(\gamma-1)$, respectively) are not affected by the presence of the generator $E_{\gamma,\gamma\pm 1}$, since for those rearrangements the difference of γ and $(\gamma+1)$ ($(\gamma-1)$, respectively) is immaterial. □

Proposition 4.2.

$$\tilde{T}_{nn}(\lambda) = \bigotimes_{i=1}^{N} \operatorname{diag}\{b(\lambda-z_i),\ldots,b(\lambda-z_i),1\}. \tag{37}$$

Proof. Let us consider the action of the matrix F on T_{nn}

$$F_{1\ldots N}T_{nn} = \sum_{\sigma\in S_N}\sum^{*}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}\prod_{i=1}^{N}P^{\sigma(i)}_{\alpha_{\sigma(i)}}R^{\sigma}_{1\ldots N}P^0_nT_{0,1\ldots N}P^0_n$$

$$= \sum_{\sigma\in S_N}\sum^{*}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}\prod_{i=1}^{N}P^{\sigma(i)}_{\alpha_{\sigma(i)}}P^0_nT_{0,\sigma(1)\ldots\sigma(N)}P^0_nR^{\sigma}_{1\ldots N}. \tag{38}$$

The specialization to the entry (n,n) of the auxiliary space is achieved here by the projectors P^0_n. For the second equality in (38) we have used relation (8) and the obvious fact that P^0_n commutes with $R^{\sigma}_{1\ldots N}$. To simplify the following argument we distinguish in the sum $\sum^*$ cases of various multiplicities of the occurrence of the group index n:

$$F_{1\ldots N}T_{nn} = \sum_{\sigma\in S_N}\sum_{k=0}^{N}\sum^{*'}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}\prod_{j=N-k+1}^{N}\delta_{\alpha_{\sigma(j)},n}P^{\sigma(j)}_n\prod_{j=1}^{N-k}P^{\sigma(j)}_{\alpha_{\sigma(j)}}P^0_nT_{0,\sigma(1)\ldots\sigma(N)}P^0_nR^{\sigma}_{1\ldots N}. \tag{39}$$

Let us consider the prefactor of $R^{\sigma}_{1\ldots N}$ on the right-hand side of equation (39) more closely. Using specific features of the R-matrices we can rewrite it as follows:

$$\prod_{j=1}^{N-k}P^{\sigma(j)}_{\alpha_{\sigma(j)}}\prod_{j=N-k+1}^{N}P^{\sigma(j)}_nP^0_n\,T_{0,\sigma(1)\ldots\sigma(N)}\,P^0_n$$

$$= \prod_{j=1}^{N-k}P^{\sigma(j)}_{\alpha_{\sigma(j)}}\left(R_{0,\sigma(N)}\right)^{nn}_{nn}\left(R_{0,\sigma(N-1)}\right)^{nn}_{nn}\ldots\left(R_{0,\sigma(N-k+1)}\right)^{nn}_{nn}$$

$$\times P^0_n\,T_{0,\sigma(1)\ldots\sigma(N-k)}\,P^0_n\prod_{j=N-k+1}^{N}P^{\sigma(j)}_n$$

$$= \prod_{j=1}^{N-k}P^{\sigma(j)}_{\alpha_{\sigma(j)}}P^0_n\,T_{0,\sigma(1)\ldots\sigma(N-k)}\,P^0_n\prod_{j=N-k+1}^{N}P^{\sigma(j)}_n$$

$$= \prod_{i=1}^{N-k}\left(R_{0\sigma(i)}\right)^{n,\alpha_{\sigma(i)}}_{n,\alpha_{\sigma(i)}}P^0_n\prod_{j=1}^{N-k}P^{\sigma(j)}_{\alpha_{\sigma(j)}}\prod_{j=N-k+1}^{N}P^{\sigma(j)}_n. \tag{40}$$

Inserting the right-hand side of (40) into equation (39) one sees that the product $\prod_i\left(R_{0\sigma(i)}\right)^{n,\alpha_{\sigma(i)}}_{n,\alpha_{\sigma(i)}}$ provides the desired diagonal dressing factor of T_{nn} and the product of projectors applied to R^{σ} restores $F_{1\ldots N}$.

This completes the proof of proposition 4.2. □

Given the simple root generators $\tilde{E}_{\alpha,\alpha\pm1}$ it is a straightforward task to evaluate the generators corresponding to non-simple roots.

One finds, in particular,

$$\tilde{E}_{n-\alpha,n} = \sum_{k=1}^{\alpha}\sum_{i_1\neq\cdots\neq i_k}\prod_{\gamma=1}^{k-1}\frac{\eta}{z_{i_\gamma}-z_{i_{\gamma+1}}}\sum_{\alpha=\beta_0>\beta_1\ldots>\beta_k=0}\bigotimes_{l=1}^{k}E^{(i_l)}_{n-\beta_{l-1},n-\beta_l}\bigotimes_{j\neq i_1\ldots i_k}\Gamma^{(j)}_{j;\underbrace{i_1\ldots i_1}_{\beta_0-\beta_1}\underbrace{i_2\ldots i_2}_{\beta_1-\beta_2}\ldots\underbrace{i_k\ldots i_k}_{\beta_{k-1}-\beta_k}} \tag{41}$$

where $\Gamma_{j;i_1,\ldots,i_\alpha} = \mathrm{diag}\{1,\ldots,1,b^{-1}_{i_1j},\ldots,b^{-1}_{i_\alpha j},1\}$.

Exploiting the last equation and equation (32) one finally arrives at

$$\tilde{T}_{n\,n-\alpha} = \sum_{k=1}^{\alpha} \sum_{i_1 \neq \cdots \neq i_k} c(\lambda - z_{i_k}) \prod_{\gamma=1}^{k-1} \frac{\eta}{z_{i_\gamma} - z_{i_{\gamma+1}}} b(\lambda - z_{i_\gamma})$$
$$\times \sum_{\alpha=\beta_0>\beta_1>\cdots>\beta_k=0} \bigotimes_{l=1}^{k} E^{(i_l)}_{n-\beta_{l-1},n-\beta_l} \bigotimes_{j\neq i_1\ldots i_k} \Delta^{(j)}_{j;\underbrace{i_1\ldots i_1}_{\beta_0-\beta_1}\underbrace{i_2\ldots i_2}_{\beta_1-\beta_2}\ldots\underbrace{i_k\ldots i_k}_{\beta_{k-1}-\beta_k}} \tag{42}$$

where $\Delta^{(j)}_{j;i_1,\ldots,i_\alpha} = \mathrm{diag}\{b(\lambda - z_j), \ldots, b(\lambda - z_j), b(\lambda - z_j)\, b^{-1}_{i_1 j}, \ldots, b(\lambda - z_j)\, b^{-1}_{i_\alpha j}, 1\}$ is a diagonal dressing matrix acting in jth space.

5. Bethe wavevectors

We give brief details concerning the description of the hierarchical Bethe ansatz and refer the reader to [9, 11] for more details.

The operators $T_{n\alpha}(\lambda)$ $(1 \leqslant \alpha < n-1)$ serve in the $sl(n)$ problem as quasiparticle creation operators and the corresponding operators $T_{\alpha n}(\lambda)$ have the role of annihilation operators.

The $T_{n\alpha}(\lambda)$ satisfy the Faddeev–Zamolodchikov algebra

$$\begin{aligned}
&[T_{n\alpha}(\lambda_1), T_{n\alpha}(\lambda_2)] = 0 \\
&T_{n\alpha}(\lambda_1)T_{n\beta}(\lambda_2) = \frac{1}{b(\lambda_2 - \lambda_1)} T_{n\beta}(\lambda_2)T_{n\alpha}(\lambda_1) - \frac{c(\lambda_2 - \lambda_1)}{b(\lambda_2 - \lambda_1)} T_{n\beta}(\lambda_1)T_{n\alpha}(\lambda_2)
\end{aligned} \tag{43}$$

where in the last relation $\alpha \neq \beta$.

An ansatz for a Bethe vector Ψ_n is given in terms of a linear superposition of products of operators $T_{n\alpha}$ acting on a reference state $\Omega^{(n)}_N$:

$$\Psi_n(N; \lambda_1, \ldots, \lambda_p) = \sum_{\alpha_1,\ldots,\alpha_p} \Phi_{\alpha_1,\ldots,\alpha_p} T_{n\alpha_1}(\lambda_1) \ldots T_{n\alpha_p}(\lambda_k)\, \Omega^{(n)}_N \tag{44}$$

where the reference state $\Omega^{(n)}_N$ is constituted as an N-fold tensor product of lowest-weight states $v^{(i)}_n$ in $\mathbb{C}^{(i)}_n$

$$\Omega_N = \bigotimes_{i=1}^{N} v^{(i)}_n$$

and the $\Phi_{\alpha_1,\ldots,\alpha_p}$ denote some c-number coefficients.

It is important to note that the reference state is invariant under the F-transformation:

$$F\, \Omega^{(n)}_N = \Omega^{(n)}_N$$

since it is immediately obvious from the definition (15) of F that from the sum over the permutation group only the term with the unit element inscribed gives a non-vanishing result when applied to $\Omega^{(n)}_N$.

It can be shown [9, 11] that Ψ_n is an eigenvector of the transfer matrix $t(\lambda) = \sum_i T_{ii}(\lambda)$ if

(a) the parameters $\lambda_1, \ldots, \lambda_p$ satisfy a certain system of rational equations (the famous Bethe ansatz equations), and if

(b) the c-number coefficients are chosen such that they constitute the components of a rational $sl(n-1)$ transfer matrix.

One establishes therewith a recursive procedure leading finally to an $sl(2)$ eigenvalue problem. We will keep the spectral parameters arising at the various stages of the procedure general, instead of specializing them to solutions of the Bethe ansatz equations. In other words, we keep the Bethe vector 'off-shell' [12]. Our goal in this paper is to figure out the functional form of the Bethe wavevectors.

To start with we recall the form of the $sl(2)$ wavevectors in the basis provided by Maillet and Sanchez de Santos [1]. The creation operators with respect to the lowest-weight reference state (in the special basis) are of the form

$$\tilde{T}_{21}(\lambda) = \sum_{i=1}^{N} c(\lambda - z_i)\sigma_+^{(i)} \bigotimes_{j\neq i} \begin{pmatrix} b(\lambda - z_j)b_{ij}^{-1} & 0 \\ 0 & 1 \end{pmatrix}_{[j]}. \tag{45}$$

The ensuing Bethe wavevectors are given by

$$\begin{aligned} \Psi_2(N; \lambda_1, \ldots, \lambda_p) &= \tilde{T}_{21}(\lambda_1) \ldots \tilde{T}_{21}(\lambda_p)\, \Omega_N^{(2)} \\ &= \sum_{i_1 \neq \cdots \neq i_p} B_p^{(2)}(\lambda_1, \ldots, \lambda_p | z_{i_1}, \ldots, z_{i_p}) \sigma_+^{(i_1)} \ldots \sigma_+^{(i_p)}\, \Omega_N^{(2)}. \end{aligned} \tag{46}$$

The c-number coefficients $B^{(2)}(\{\lambda_i\}|\{z_i\})$ of the last equation can easily be worked out, taking into account the 'diagonal dressing' factors of the spin raising operators σ_+^i in (45), to be of the form

$$B_p^{(2)}(\lambda_1, \ldots, \lambda_p | z_1, \ldots, z_p) = \sum_{\sigma \in S_p} \prod_{m=1}^{p} c(\lambda_m - z_{\sigma(m)}) \prod_{l=m+1}^{p} \frac{b(\lambda_m - z_{\sigma(l)})}{b(z_{\sigma(m)} - z_{\sigma(l)})}. \tag{47}$$

A concise alternative representation of the coefficients $B_p^{(2)}$ has been derived in [4]:

$$B_p^{(2)}(\lambda_1, \ldots, \lambda_p | z_{i_1}, \ldots, z_{i_p}) = \frac{\prod_{i,j}(\lambda_i - z_j)}{\prod_{i>j}(\lambda_i - \lambda_j)(z_j - z_i)} \det_{\langle i,j\rangle} \left(\frac{1}{\lambda_i - z_j} - \frac{1}{\lambda_i - z_j + \eta} \right). \tag{48}$$

The vectors $\Psi_p^{(2)}(N; \lambda_1, \ldots, \lambda_p)$ are invariant under arbitrary exchanges of the variables $\lambda_1, \ldots, \lambda_p$ since operators $\tilde{T}_{21}$ with different values of the attached spectral parameters do commute with each other.

It has been shown in [11, 13] that this symmetric appearance of the spectral parameters in the wavevectors $\Psi^{(n)}$, $n > 2$ persists, despite the Faddeev–Zamolodchikov relations (43), under the assumption that the coefficients $\Phi_{\alpha_1,\ldots,\alpha_p}$ in (44) are components of an $sl(n-1)$ Bethe wavevector. Our argument below will rely heavily on this exchange symmetry.

We now discuss the $sl(3)$ model. The generalization to $sl(n)$, $n > 3$ will subsequently be rather obvious. Equation (42) specialized to the case of $sl(3)$ renders the creation operators in the F-basis as

$$\tilde{T}_{32} = \sum_{i=1}^{N} c(\lambda - z_i) E_{23}^{(i)} \bigotimes_{j\neq i} \mathrm{diag}\{b(\lambda - z_j), b(\lambda - z_j)b_{ij}^{-1}, 1\}_{[j]} \tag{49}$$

$$\begin{aligned} \tilde{T}_{31} = &\sum_{i=1}^{N} c(\lambda - z_i) E_{13}^{(i)} \bigotimes_{j\neq i} \mathrm{diag}\{b(\lambda - z_j)b_{ij}^{-1}, b(\lambda - z_j)b_{ij}^{-1}, 1\}_{[j]} \\ &+ \sum_{i\neq j} c(\lambda - z_i) b(\lambda - z_j) \frac{\eta}{z_i - z_j} E_{23}^{(i)} \otimes E_{12}^{(j)} \\ &\times \bigotimes_{k\neq i,j} \mathrm{diag}\{b(\lambda - z_k)b_{jk}^{-1}, b(\lambda - z_k)b_{ik}^{-1}, 1\}_{[k]}. \end{aligned} \tag{50}$$

The strategy employed in determining the form of the Bethe wavevector (44) will be as follows. We select a particular order in which the operators $T_{n\alpha}$ act on the reference state such that the eventual explicit evaluation becomes as simple as possible. (This particular order can always be achieved by use of the Faddeev–Zamolodchikov relations (43).) The c-number coefficient $\Phi^{(2)}$ has to be taken in the original basis and not in the F basis, but fortunately a particular coefficient in the sum (44) (specialized to $sl(3)$) is invariant under the similarity transformation induced by the F-matrices. This enables one to compute the explicit form of this special coefficient $\Phi^{(2)}$ using the result (47) and relate it to the order of operators alluded to in the preceding point by an appropriate factor $\prod_{ij} b^{-1}(\lambda_i - \mu_j)$. One uses the permutation symmetry to determine all other terms in the sum.

Following this line of thought we arrive at the following.

Proposition 5.1.

$$\tilde{\Psi}_3(N, \lambda_1, \ldots, \lambda_{p_0}; \lambda_{p_0+1}, \ldots, \lambda_{p_0+p_1}) = \sum_{\sigma \in S_{p_0}} B^{(2)}_{p_1}(\lambda_{p_0+1}, \ldots, \lambda_{p_0+p_1} | \lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(p_1)})$$
$$\times \prod_{k=1}^{p_1} \prod_{l=p_1+1}^{p_0} b(\lambda_{\sigma(k)} - \lambda_{\sigma(l)})^{-1} \tilde{T}_{32}(\lambda_{\sigma(p_1+1)}) \ldots \tilde{T}_{32}(\lambda_{\sigma(p_0)})$$
$$\times \tilde{T}_{31}(\lambda_{\sigma(1)}) \ldots \tilde{T}_{31}(\lambda_{\sigma(p_1)})\, \Omega_N^{(3)}. \tag{51}$$

Proof. The proof of this formula proceeds as mentioned above.

We have specialized the form of the ansatz in equation (51) as compared with equation (44) so that operators $\tilde{T}_{32}$ are placed to the left of all operators $\tilde{T}_{31}$. The latter order can be achieved by moving the operators $\tilde{T}_{32}$ in the general ansatz (44) to the required position with the help of the Faddeev–Zamolodchikov relations (43). Let us consider, in particular, the vector contributing in (44) of the form

$$\tilde{T}_{31}(\lambda_1) \ldots \tilde{T}_{31}(\lambda_{p_1}) \tilde{T}_{32}(\lambda_{p_1+1}) \ldots \tilde{T}_{32}(\lambda_{p_0})\, \Omega_N^{(3)} \tag{52}$$

and let us relate it to the vector contributing in (51) of the form

$$\tilde{T}_{32}(\lambda_{p_1+1}) \ldots \tilde{T}_{32}(\lambda_{p_0}) \tilde{T}_{31}(\lambda_1) \ldots \tilde{T}_{31}(\lambda_{p_1})\, \Omega_N^{(3)}. \tag{53}$$

A diligent appreciation of the Faddeev–Zamolodchikov relations leads to the conclusion that (53) has its unique origin in (52) and that, moreover, only the first term on the right-hand side of (43) supplies contributions in the transition from (52) to (53). It follows that the transition from (52) to (53) is accompanied by an additional factor

$$\prod_{x=1}^{p_1} \prod_{y=p_1+1}^{p_0} \frac{1}{b(\lambda_x - \lambda_y)}. \tag{54}$$

The c-number coefficients $\Phi_{\alpha_1 \ldots \alpha_p}$ in (44) (when specialized to the case $n = 3$) refer to an $sl(2)$ Bethe wavevector in the familiar basis commonly used for the algebraic Bethe ansatz—not that of Maillet and Sanchez de Santos.

However, we want to argue that the special coefficient $\Phi^{(2)}_{1\ldots12\ldots2}$ (the factor which accompanies the vector (53)) is, in fact, the same in both frames. One has to note first that the similarity transformation by the F-matrices (specialized to the case of $sl(3)$) respects the $sl(2)$ structure. This means among other things that components only with the same number of labels 1 and 2 are related to each other through the similarity transformation. One has secondly to observe that in the transformation of $\Phi^{(2)}_{1\ldots12\ldots2}$ no other components with a different order of labels can appear due to the lower triangularity of F. (The matrix F would otherwise not be lower triangular.)

One finds thirdly through a direct examination of the definition of F that its diagonal elements relating the coefficients $\Phi^{(2)}_{1\ldots12\ldots2}$ in the two frames to each other are equal to unity. Therefore, we know the coefficient $\Phi^{(2)}_{1\ldots12\ldots2}$ to be of the Maillet–Sanchez de Santos form.

Invoking the above-mentioned exchange symmetry one completes the proof. □

The expression (51) for $\tilde{\Psi}_3$ can be worked out further by inserting the definitions (50) and (49) of $\tilde{T}_{31}$ and $\tilde{T}_{32}$, respectively, to yield

$$\begin{aligned}\tilde{\Psi}_3(N,\lambda_1,\ldots,\lambda_{p_0};\lambda_{p_0+1},\ldots,\lambda_{p_0+p_1})\\ &= \sum_{i_1\neq\cdots\neq i_{p_0}} B^{(3)}_{p_0,p_1}(\lambda_1,\ldots,\lambda_{p_0};\lambda_{p_0+1},\ldots,\lambda_{p_0+p_1}|z_{i_1},\ldots,z_{i_{p_0}})\\ &\quad\times E_{23}^{(i_{p_1+1})}\ldots E_{23}^{(i_{p_0})}E_{13}^{(i_1)}\ldots E_{13}^{(i_{p_1})}\Omega_N^{(3)}.\end{aligned} \tag{55}$$

The order of operators adopted in equation (51) yields the bonus that the second term on the right-hand side (the twofold sums) of (50) do not appear in (55), since those drop out if applied to the reference state Ω_N.

The sets of operators $\tilde{T}_{31}$ and $\tilde{T}_{32}$ generate through their respective diagonal dressing the structure of two $sl(2)$ wavevectors together with a factor which accounts for the way in which the operators $\tilde{T}_{32}$ respond to operators $\tilde{T}_{31}$ on their right-hand side (cf equation (52)).

This completes our goal to reduce the $sl(3)$ Bethe wavevectors to $sl(2)$ structures:

$$\begin{aligned}B^{(3)}_{p_0,p_1}(\lambda_1,\ldots,\lambda_{p_0};\lambda_{p_0+1},\ldots,\lambda_{p_0+p_1}|z_{i_1},\ldots,z_{i_{p_0}})\\ &= \sum_{\sigma\in S_{p_0}}\prod_{k=1}^{p_1}\prod_{l=p_1+1}^{p_0}\frac{b(\lambda_{\sigma(l)}-z_{i_k})}{b(\lambda_{\sigma(k)}-\lambda_{\sigma(l)})}B^{(2)}_{p_0-p_1}(\lambda_{\sigma(p_1+1)},\ldots,\lambda_{\sigma(p_0)}|z_{i_{p_1+1}},\ldots,z_{i_{p_0}})\\ &\quad\times B^{(2)}_{p_1}(\lambda_{p_0+1},\ldots,\lambda_{p_0+p_1}|\lambda_{\sigma(1)},\ldots,\lambda_{\sigma(p_1)})B^{(2)}_{p_1}(\lambda_{\sigma(1)},\ldots,\lambda_{\sigma(p_1)}|z_{i_1},\ldots,z_{i_{p_1}}).\end{aligned} \tag{56}$$

All ingredients of our argument for the case of $sl(3)$ can be straightforwardly generalized to $sl(n)$; $n>3$.

We collect all operators $\tilde{T}_{n\,n-\alpha}$ to the left of operators $\tilde{T}_{n\,n-\beta}$ if $\alpha<\beta$. Once again only the first term in the expression (42) of the respective operators $\tilde{T}_{n\,n-i}$ contributes in this special ordering.

The wavefunction $\tilde{\Psi}_n$ is then expressed in analogy to equation (55) by

$$\begin{aligned}\tilde{\Psi}_n(N,p_0,p_1,\ldots,p_{n-2}) &= \sum_{i_1\neq\cdots\neq i_{p_0}} B^{(n)}_{p_0,p_1,\ldots,p_{n-2}}(\lambda_1,\ldots,\lambda_{p_0+\cdots p_{n-2}}|z_{i_1},\ldots,z_{i_{p_0}})\\ &\quad\times\prod_{\alpha=1}^{n-1}\prod_{j=p_\alpha+1}^{p_{\alpha-1}}E^{(i_j)}_{n-\alpha\,n}\Omega_N^{(n)}\end{aligned} \tag{57}$$

with the following recursion relation for the function $B^{(n)}$:

$$\begin{aligned}B^{(n)}_{p_0\,p_1\,\ldots\,p_{n-2}}(\lambda_1,\ldots,\lambda_{p_0+p_1+\cdots p_{n-2}}|z_1,\ldots,z_{p_0})\\ &= \sum_{\sigma\in S_{p_0}}\prod_{\alpha=1}^{n-2}\prod_{k_\alpha=p_{\alpha+1}+1}^{p_\alpha}\prod_{l_\alpha=p_\alpha+1}^{p_0}\frac{b(\lambda_{\sigma(l_\alpha)}-z_{k_\alpha})}{b(\lambda_{\sigma(k_\alpha)}-\lambda_{\sigma(l_\alpha)})}\\ &\quad\times\prod_{\gamma=0}^{n-2}B^{(2)}_{p_\gamma-p_{\gamma+1}}(\lambda_{\sigma(p_{\gamma+1}+1)}\ldots\lambda_{\sigma(p_\gamma)}|z_{p_{\gamma+1}+1}\ldots z_{p_\gamma})\\ &\quad\times B^{(n-1)}_{p_1\cdots p_{n-2}}(\lambda_{p_0+1}\ldots\lambda_{p_0+p_1+\cdots+p_{n-2}}|\lambda_{\sigma(1)}\ldots\lambda_{\sigma(p_1)}).\end{aligned} \tag{58}$$

The resolution of the recursion gives

$$
\begin{aligned}
&B^{(n)}_{p_0\,p_1\,\ldots\,p_{n-2}}(\lambda_1,\ldots,\lambda_{p_0+p_1+\cdots+p_{n-2}}|z_1,\ldots,z_{p_0}) \\
&\quad= \sum_{\sigma_0\in S_{p_0}}\sum_{\sigma_1\in S_{p_1}}\cdots\sum_{\sigma_{n-3}\in S_{p_{n-3}}}\prod_{i=0}^{n-2}\prod_{\alpha_i=i+1}^{n-2}\prod_{k_{\alpha_i}=p_{\alpha_i+1}+1}^{p_{\alpha_i}}\prod_{l_{\alpha_i}=p_{\alpha_i}+1}^{p_i} \\
&\quad\times\frac{b(\lambda_{q_{i-1}+\sigma_i(l_{\alpha_i})}-\lambda_{\sigma_{i-1}(k_{\alpha_i})})}{b(\lambda_{q_{i-1}+\sigma_i(k_{\alpha_i})}-\lambda_{q_{i-1}+\sigma_i(l_{\alpha_i})})} \\
&\quad\times\prod_{\gamma_i=i}^{n-2}B^{(2)}_{p_{\gamma_i}-p_{\gamma_i+1}}\left(\lambda_{q_{i-1}+\sigma_i(p_{\gamma_{i+1}+1})}\ldots\lambda_{q_{i-1}+\sigma_i(p_{\gamma_i})}|\lambda_{\sigma_{i-1}(p_{\gamma_{i+1}+1})}\ldots\lambda_{\sigma_{i-1}(p_{\gamma_i})}\right) \\
&\quad\times B^{(2)}_{p_{n-2}}\left(\lambda_{q_{n-3}+1}\ldots\lambda_{q_{n-3}+p_{n-2}}|\lambda_{q_{n-4}+\sigma_{n-3}(1)}\ldots\lambda_{q_{n-4}+\sigma_{n-3}(p_{n-2})}\right) \qquad (59)
\end{aligned}
$$

where by definition

$$q_i=\sum_{j=0}^{i}p_j \qquad q_{-1}=0$$

and

$$\lambda_{\sigma_{-1}(k)}=z_k.$$

Equations (57) and (59) supply the explicit representation of the $sl(n)$ wavevectors in terms of $sl(2)$ vectors, that is, the resolution of the Bethe hierarchy.

6. Conclusions

The form of the factorizing F-matrix presented in section 3 is of an intriguing simplicity. We suspect that a representation theoretical aspect is lurking behind it which escapes our present knowledge. It should be noted that we arrived at this ansatz by guesswork immediately for the full F-matrix instead of taking the detour via partial F-matrices, as proposed in [1]. It seems rather likely that we would have missed the simplicity of the ansatz if we had chosen the approach via partial F-matrices.

Our original hope was to find a structure for the Bethe wavevectors which is as suggestive as that displayed for the case of Gaudin magnets in [10]. This goal has not yet been achieved completely since we are not in possession of an entirely satisfactory representation of $sl(2)$ wavevectors, which are the building blocks for the final formula (59) of section 5. The representations (47) and (48) both have the drawback that they do display the singularity structure of the wavevectors in a redundant manner. (The matter is discussed further in appendix B.) We, nevertheless, nourish the hope that our findings will be of some help in bringing effective large-n calculations of thermodynamical quantities within the range of the algebraic Bethe ansatz method.

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Appendix A

In this appendix we verify the $sl(n)$ algebra relations, taking the formulae for the generators $\tilde{E}_{\alpha,\alpha\pm1}$ of section 4 as a starting point.

We use the following defining relations for a semisimple Lie algebra [14].

Fix a root system with a basis $\{\alpha_1, \ldots, \alpha_l\}$. Let L be the Lie algebra generated by three l elements $\{E_{+\alpha_i}, E_{-\alpha_i}, H_i;\ 1 \leqslant i \leqslant l\}$. L is uniquely determined by the relations

S1 $\quad \left[E_{+\alpha_i}, E_{-\alpha_j}\right] = \delta_{ij} H_{\alpha_i}$

S2 $\quad \left[H_{\alpha_i}, E_{\pm\alpha_j}\right] = \pm A_{ji} E_{\pm\alpha_j}$

S3 $\quad \left[H_{\alpha_i}, H_{\alpha_j}\right] = 0$

S4† $\quad \left(ad_{E^{\alpha_i}_{\pm}}\right)^{1-A_{ji}} \left(E^{\alpha_j}_{\pm}\right) = 0 \qquad i = 1, \ldots, l \quad i \neq j$

with $A_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ denoting the Cartan matrix.

We recall from section 4 the expressions for the generators of the algebra $sl(n)$ corresponding to simple roots:

$$\begin{aligned}
\tilde{E}_{+\alpha} &= \sum_{i=1}^{N} E^{(i)}_{+\alpha} \bigotimes_{j\neq i} \left(\mathbb{1}_N + \frac{\eta}{z_i - z_j} e_{\alpha\,\alpha}\right)_{[j]} \equiv \sum_{i=1}^{N} E^{(i)}_{+\alpha} \bigotimes_{j\neq i} \triangle^{\alpha}_{(i,j)} \\
\tilde{E}_{-\alpha} &= \sum_{i=1}^{N} E^{(i)}_{-\alpha} \bigotimes_{j\neq i} \left(\mathbb{1}_N + \frac{\eta}{z_j - z_i} e_{\alpha+1\,\alpha+1}\right)_{[j]} \equiv \sum_{i=1}^{N} E^{(i)}_{-\alpha} \bigotimes_{j\neq i} \tilde{\triangle}^{\alpha}_{(j,i)}
\end{aligned} \tag{A1}$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ are the elementary matrices and $\left(E^{(k)}_{+\alpha}\right)_{ij} = \delta_{\alpha\,i}\delta_{\alpha+1\,j}$, $\left(E^{(k)}_{-\alpha}\right)_{ij} = \delta_{\alpha+1\,i}\delta_{\alpha\,j}$ are the simple roots of $sl(n)$ acting in the kth space.

Using their definitions one has‡

$$\begin{aligned}
\left[\tilde{E}_{+\alpha}, \tilde{E}_{-\beta}\right] &= \sum_i \left[E^{(i)}_{+\alpha}, E^{(i)}_{-\beta}\right] \bigotimes_{j\neq i} \triangle^{\alpha}_{(i,j)} \tilde{\triangle}^{\beta}_{(j,i)} \\
&\quad + \sum_{i,j}{}' \left(E^{(i)}_{\alpha} \tilde{\triangle}^{-\beta}_{(i,j)} \otimes \triangle^{\alpha}_{(i,j)} E^{(j)}_{-\beta} - \tilde{\triangle}^{-\beta}_{(i,j)} E^{(i)}_{\alpha} \otimes E^{(j)}_{-\beta} \triangle^{\alpha}_{(i,j)}\right) \bigotimes_{k\neq i,j} \triangle^{\alpha}_{(i,k)} \tilde{\triangle}^{-\beta}_{(j,k)} \\
&= \sum_i \delta_{\alpha\,\beta} H^{(i)}_{\alpha} \bigotimes_{j\neq i} \triangle^{\alpha}_{(i,j)} \tilde{\triangle}^{\alpha}_{(j,i)}
\end{aligned} \tag{A2}$$

where we have exploited the fact that the second sum vanishes term by term identically for all $\{\alpha, -\beta\}$.

The dressing can be written as $\triangle^{\alpha}_{(i,j)} \tilde{\triangle}^{\alpha}_{(j,i)} = \mathbb{1}_{[j]} + \frac{\eta}{z_i - z_j} H^{(j)}_{\alpha}$, because $(H_\alpha)_{ij} = \delta_{\alpha\,i}\delta_{\alpha\,j} - \delta_{\alpha+1\,i}\delta_{\alpha+1\,j}$.

Against to first appearance the Cartan operators H_α remain without dressing. For this purpose we consider the expression

$$\bigotimes_{i=1}^{N} \left(\mathbb{1}_{[i]} + \frac{\eta}{\lambda - z_i} H^{(i)}_{\alpha}\right) = \mathbb{1}_{[N]} + \sum_{i=1}^{N} \frac{\eta}{\lambda - z_i} H^{(i)}_{\alpha} \bigotimes_{j\neq i} \left(\mathbb{1}_{[j]} + \frac{\eta}{z_i - z_j} H^{(j)}_{\alpha}\right). \tag{A3}$$

This identity can be proved by noting that both sides have the same limit for $\lambda \to \infty$ and that the residues at the simple poles $\lambda = z_i$ are identical. If we now consider the order $1/\lambda$ in the

† $(ad_x)^n$ is a shorthand notation for $\underbrace{ad_x \circ ad_x \circ \cdots \circ ad_x}_{n \text{ times}}$, such that, for example, $(ad_x)^2(y) = [x, [x, y]]$.

‡ $\sum'_{i,j}$ means $\sum_{i,j\ i\neq j}$.

expansion of both sides we obtain

$$\sum_i H_\alpha^{(i)} \bigotimes_{j\neq i} \mathbb{1}_{[j]} = \sum_i H_\alpha^{(i)} \bigotimes_{j\neq i} \left(\mathbb{1}_{[j]} + \frac{\eta}{z_i - z_j} H_\alpha^{(j)} \right) = \sum_i H_\alpha^{(i)} \bigotimes_{j\neq i} \triangle^\alpha_{(i,j)} \tilde{\triangle}^\alpha_{(j,i)} \tag{A4}$$

which concludes the proof that the Cartan operators associated with the simple roots acquire no dressing, which in turn renders the proof of the commutativity of the Cartan operators trivial. To prove the Serre relation

$$\left(ad_{E_\pm^{\alpha^i}}\right)^{1-A_{ji}} \left(E_\pm^{\alpha^j}\right) = 0 \qquad i = 1, \ldots, N-1; \quad i \neq j \tag{A5}$$

we have to distinguish two cases:

$$\begin{array}{lll} \text{(a)} & |j-i| = 1 & \left[E_\pm^{\alpha^i}, \left[E_\pm^{\alpha^i}, E_\pm^{\alpha^j}\right]\right] = 0 \\ \text{(b)} & |j-i| > 1 & \left[E_\pm^{\alpha^i}, E_\pm^{\alpha^j}\right] = 0 \end{array} \tag{A6}$$

as all other matrix elements of the Cartan matrix are zero. (For $sl(n)$ we have $A_{ii} = 2$, $A_{i+1\,i} = A_{i\,i+1} = -1$ and $A_{ij} = 0$ otherwise.)

To proceed with the proof we list some useful relations:

$$\begin{aligned} \left(E_{+\alpha}\, \triangle^\beta_{(i,j)}\right) &= d^\beta_{\alpha+1}(i,j)\ E_{+\alpha} \\ \left(\triangle^\beta_{(i,j)}\, E_{+\alpha}\right) &= d^\beta_{\alpha}(i,j)\ E_{+\alpha} \\ \left(E_{-\alpha}\, \triangle^\beta_{(i,j)}\right) &= d^\beta_{\alpha}(i,j)\ E_{-\alpha} \\ \left(\triangle^\beta_{(i,j)}\, E_{-\alpha}\right) &= d^\beta_{\alpha+1}(i,j)\ E_{-\alpha} \end{aligned} \tag{A7}$$

where $d^\beta_\alpha(i,j)$ means the αth element on the diagonal of the matrix $\triangle^\beta_{(i,j)}$.

We now look at the first case of (A6) and show the argument for the positive roots:

$$\left[\tilde{E}_\alpha, \tilde{E}_\beta\right] = \sum_i \left[E_\alpha^{(i)}, E_\beta^{(i)}\right] \bigotimes_{j\neq i} \triangle^\alpha_{(i,j)} \triangle^\beta_{(i,j)} + \sum_{i,j}{}' \frac{\eta}{z_j - z_i} E_\alpha^{(i)} \otimes E_\beta^{(j)} \bigotimes_{k\neq i,j} \triangle^\alpha_{(i,k)} \triangle^\beta_{(j,k)} \tag{A8}$$

and thus

$$\begin{aligned} \left[\tilde{E}_\alpha, \left[\tilde{E}_\alpha, \tilde{E}_\beta\right]\right] &= \sum_i \left[E_\alpha^{(i)}, \left[E_\alpha^{(i)}, E_\beta^{(i)}\right]\right] \bigotimes_{j\neq i} \triangle^\alpha_{(i,j)} \triangle^\alpha_{(i,j)} \triangle^\beta_{(i,j)} \\ &\quad + \sum_{i,j}{}' \Big(E_\alpha^{(i)} \triangle^\alpha_{(j,i)} \triangle^\beta_{(j,i)} \otimes \triangle^\alpha_{(i,j)} \left[E_\alpha^{(j)}, E_\beta^{(j)}\right] \\ &\quad - \triangle^\alpha_{(j,i)} \triangle^\beta_{(j,i)} E_\alpha^{(i)} \otimes \left[E_\alpha^{(j)}, E_\beta^{(j)}\right] \triangle^\alpha_{(i,j)} \Big) \bigotimes_{k\neq i,j} \triangle^\alpha_{(i,k)} \triangle^\alpha_{(j,k)} \triangle^\beta_{(j,k)} \\ &\quad + \sum_{i,j}{}' \frac{\eta}{z_j - z_i} \left(E_\alpha^{(i)} E_\alpha^{(i)} \otimes \triangle^\alpha_{(i,j)} E_\beta^{(j)} - E_\alpha^{(i)} E_\alpha^{(i)} \otimes E_\beta^{(j)} \triangle^\alpha_{(i,j)} \right) \\ &\quad \times \bigotimes_{k\neq i,j} \triangle^\alpha_{(i,k)} \triangle^\alpha_{(j,k)} \triangle^\beta_{(j,k)} + \sum_{i,j}{}' \frac{\eta}{z_j - z_i} \Big(\triangle^\alpha_{(j,i)} E_\alpha^{(i)} \otimes E_\alpha^{(j)} E_\beta^{(j)} \\ &\quad - E_\alpha^{(i)} \triangle^\alpha_{(j,i)} \otimes E_\beta^{(j)} E_\alpha^{(j)} \Big) \bigotimes_{k\neq i,j} \triangle^\alpha_{(i,k)} \triangle^\alpha_{(j,k)} \triangle^\beta_{(j,k)} \\ &\quad + \sum_{i,j,k}{}' \frac{\eta}{z_k - z_j} \Big(E_\alpha^{(i)} \triangle^\alpha_{(j,i)} \triangle^\beta_{(k,i)} \otimes \triangle^\alpha_{(i,j)} E_\alpha^{(j)} \otimes \triangle^\alpha_{(i,k)} E_\beta^{(k)} \\ &\quad - \triangle^\alpha_{(j,i)} \triangle^\beta_{(k,i)} E_\alpha^{(i)} \otimes E_\alpha^{(j)} \triangle^\alpha_{(i,j)} \otimes E_\beta^{(k)} \triangle^\alpha_{(i,k)} \Big) \bigotimes_{l\neq i,j,k} \triangle^\alpha_{(i,l)} \triangle^\alpha_{(j,l)} \triangle^\beta_{(k,l)}. \end{aligned} \tag{A9}$$

The first term in this sum vanishes due to the Serre relation for the undressed operators, and the third because $E_\alpha E_\alpha = 0$.

The second and the fourth terms cancel each other, while the last term vanishes for fixed k, since the bracket yields

$$\frac{\eta}{z_k - z_j}\left(\left(1+\frac{\eta}{z_k - z_i}\right)\left(1+\frac{\eta}{z_i - z_j}\right) - \left(1+\frac{\eta}{z_j - z_i}\right)\right) E_\alpha^{(i)} \otimes E_\alpha^{(j)} \otimes E_\beta^{(k)} \tag{A10}$$

which is antisymmetric under the exchange of i and j.

The second case of (A6) yields

$$\begin{aligned}[\tilde{E}_\alpha, \tilde{E}_\beta] &= \sum_i [E_\alpha^{(i)}, E_\beta^{(i)}] \bigotimes_{j\neq i} \triangle^\alpha_{(i,j)} \triangle^\beta_{(i,j)} \\ &\quad + {\sum_{i,j}}' \left(E_\alpha^{(i)} \triangle^\beta_{(j,i)} \otimes \triangle^\alpha_{(i,j)} E_\beta^{(j)} - \triangle^\beta_{(j,i)} E_\alpha^{(i)} \otimes E_\beta^{(j)} \triangle^\alpha_{(i,j)}\right) \bigotimes_{k\neq i,j} \triangle^\alpha_{(i,k)} \triangle^\beta_{(j,k)}\end{aligned} \tag{A11}$$

where the first term in the sum vanishes due to the assumption for the undressed operators and the second term vanishes as the bracket is zero for $|\alpha - \beta| > 1$.

The proof for the $\tilde{E}_-^{\alpha^i}$ proceeds along the same lines.

We proceed to give the form of the non-simple roots which can be obtained as multiple commutators of simple roots (proof by induction on α)

$$\begin{aligned}\tilde{E}_{i-\alpha\, i} &= [\tilde{E}_{i-\alpha\, i-\alpha+1}, \ldots, [\tilde{E}_{i-3\, i-2}, [\tilde{E}_{i-2\, i-1}, \tilde{E}_{i-1\, i}]] \ldots] \\ &= \sum_{k=1}^{\alpha} \sum_{i_1\neq\cdots\neq i_k} \prod_{\gamma=1}^{k-1} \frac{\eta}{z_{i_\gamma} - z_{i_{\gamma+1}}} \sum_{\alpha=\beta_0>\beta_1>\cdots>\beta_k=0} \bigotimes_{l=1}^{k} E^{(i_l)}_{i-\beta_{l-1},\, i-\beta_l} \\ &\quad \times \bigotimes_{j\neq i_1\ldots i_k} \Gamma^{(j)}_{\underbrace{i_k\ldots i_k}_{\beta_0-\beta_1} \underbrace{i_{k-1}\ldots i_{k-1}}_{\beta_1-\beta_2} \ldots \underbrace{i_1\ldots i_1}_{\beta_{k-1}-\beta_k};j;i}\end{aligned} \tag{A12}$$

with $\Gamma^{(j)}_{j;i_k\ldots i_k i_{k-1}\ldots i_{k-1}\ldots i_1\ldots i_1;i} = \mathrm{diag}\{1, \ldots, 1, b^{-1}_{i_k j}, \ldots, b^{-1}_{i_1 j}, \underbrace{1, \ldots, 1}_{i}\}_{[j]}$.

A similar formula holds for the negative roots

$$\begin{aligned}\tilde{E}_{i\, i-\alpha} &= \sum_{k=1}^{\alpha} \sum_{i_1\neq\cdots\neq i_k} \prod_{\gamma=1}^{k-1} \frac{\eta}{z_{i_\gamma} - z_{i_{\gamma+1}}} \sum_{\alpha=\beta_0>\beta_1>\cdots>\beta_k=0} \bigotimes_{l=1}^{k} E^{(i_l)}_{i-\beta_l,\, i-\beta_{l-1}} \\ &\quad \times \bigotimes_{j\neq i_1\ldots i_k} \Gamma^{(j)}_{\underbrace{i_k\ldots i_k}_{\beta_0-\beta_1} \underbrace{i_{k-1}\ldots i_{k-1}}_{\beta_1-\beta_2} \ldots \underbrace{i_1\ldots i_1}_{\beta_k-1-\beta_k};j;i}\end{aligned} \tag{A13}$$

with $\Gamma^{(j)}_{j;i_k\ldots i_k i_{k-1}\ldots i_{k-1}\ldots i_1\ldots i_1;i} = \mathrm{diag}\{1, \ldots, 1, b^{-1}_{j i_k}, \ldots, b^{-1}_{j i_1}, \underbrace{1, \ldots, 1}_{i-1}\}_{[j]}$.

Appendix B

In this appendix we discuss further details of the structure of the coefficients (cf equation (48))

$$\begin{aligned}B_p^{(2)}(\lambda_1, \ldots, \lambda_p; z_{i_1}, \ldots, z_{i_p}) &= \frac{\prod_{ij}(\lambda_i - z_j)}{\prod_{i>j}(\lambda_i - \lambda_j)\prod_{i>j}(z_j - z_i)} \det X \\ X_{ij} &= \frac{1}{\lambda_i - z_j} - \frac{1}{\lambda_i - z_j + \eta}.\end{aligned} \tag{B1}$$

This representation, as concise as it is, has the disadvantage that it does not reflect the singularity structure in an economic way (the poles in the prefactor on the right-hand side of (B1) are cancelled by zeros in the determinant).

One may cure the defect by appropriate manipulations on the determinant in (B1). Subtracting, for example, the last row of X from all the others, extracting a rational factor from the nth row and proceeding in the same spirit with the $(n-1)$th row and consecutively other rows one arrives at the equality

$$\begin{aligned} B_p^{(2)}(\lambda_1, \ldots, \lambda_p; z_{i_1}, \ldots, z_{i_p}) &= \frac{1}{\prod_{i<j}(z_j - z_i)} \frac{1}{\prod_{ij}(\lambda_i - z_j + \eta)} \det Y \\ Y_{\alpha,x} &= P_\alpha(\lambda; z_{p_x}) \\ P_\alpha(\lambda; z_{p_x}) &= \left\{ \prod_{i=0}^{n-\alpha}(\lambda_{n-i} - z_{p_x} + \eta) - \prod_{i=0}^{n-\alpha}(\lambda_{n-i} - z_{p_x}) \right\} \prod_{j=1}^{\alpha-1}(\lambda_j - z_{p_x} + \eta)(\lambda_j - z_{p_x}). \end{aligned} \tag{B2}$$

One should note that the polynomial P_α depends on all λ-variables but only on a single z variable. It follows that one can continue to extract polynomial factors from the determinant in (B2) by subtraction of columns from columns. The ensuing differences $P_\alpha(\lambda, z_{p_x}) - P_\alpha(\lambda, z_{p_y})$ supply the desired factors $(z_{p_x} - z_{p_y})$ which compensate the pole factors $\frac{1}{\prod_{i>j}(z_i - z_j)}$ in (B2).

Unfortunately, we have not found a concise closed form for the polynomial multiplying the remaining prefactor $\frac{1}{\prod_{ij}(\lambda_i - z_j + \eta)}$ in the final expression.

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